UDC 533.6.013.2

ANALYSIS OF ONE-DIMENSIONAL FLOW STABILITY IN A CHANNEL WITH ARBITRARY VARIATION OF STATIONARY FLOW PARAMETERS BETWEEN THE CLOSING SHOCK CROSS SECTION AND CHANNEL OUTLET

PMM Vol.41, №4, 1977, pp.637-645

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(Received April 2, 1976)

A method is developed for investigating the flow stability of perfect gas in a channel in which transition through sonic speed takes place in the so-called closing shock(*).Linearized equations which define one-dimensional nonstationary flow in a variable area channel are used. Unlike the method developed in [1-3] no additional assumptions are made about small variation of stationary flow parameters along the section between the closing shock and the channel outlet.

The proposed method is a certain modification of the method of " D -separation" [4, 5] which was already used in [2,3]. Unlike in [2,3] the construction of Nyquist is based on numerical integration of equations which define the propagation of harmonic perturbations along the channel, and, also, on the asymptotic representation of solutions of these equations for high frequencies. For the determination of the stability region results of the analysis of the characteristic asymptotic equation and considerations of continuous passing to cases investigated in [1-3] are used in addition to that method. Possibilities of the developed method are illustrated by examples of determination of boundaries of the flow stability region in the plane of coefficients of acoustic and entropy waves reflection from the cross section of the channel outlet. Comparison is made with similar results of "quasi-cylindrical" [1,2] and "transonic" [3] approximations which shows effectiveness of the latter under conditions of their applicability. We note, incidentally, that the additional assumption about slowly varying parameters of a stationary flow [1-3] not only substantially simplifies the analysis, but makes it also possible to obtain the basic controlling parameters that affect stability. Thus, for instance, in the quasi-cylindrical approximation [1,2] the effect of the channel shape manifests itself only by its angle of opening (or contraction) in the closing shock section. The "frozen coefficients" approximation is furthermore proposed and tested. It is based on the substitution in equations that define perturbation propagation along the channel of constants for the coefficients which depend on the longitudinal coordinate. The constants are obtained by averaging over the channel length.

1. Let us consider the problem of stability of one-dimensional stationary flow of perfect gas in a channel of varying cross section when the flow at channel inlet is supersonic and at its outlet subsonic. Transition through the speed of sound occurs in the

*) Editor's Note. Also called "rear shock wave".

"closing" shock at cross section where $x = 0$ with x the variable measured along the channel axis in the direction of flow. At the channel outlet plane (where $x = 1$) the condition of reflection is formulated as the linear dependence between parameters which define the incoming and outgoing waves at that plane.

According to [2,3] the analysis of stability of such flow reduces in the case of fairly considerable times t to the analysis of evolution of the following boundary value problem:

$$\frac{\partial R}{\partial t} + (U + A)\frac{\partial R}{\partial x} = a_{11}R + a_{12}L + a_{13}S \tag{1.1}$$
$$\frac{\partial L}{\partial t} + (U - A)\frac{\partial L}{\partial x} = a_{21}R + a_{22}L + a_{23}S, \quad \frac{\partial S}{\partial t} + U\frac{\partial S}{\partial x} = 0$$
$$R_+ = \varphi L_+ - \psi Y x_s, \quad S_+ = \varphi' L_+ - \psi' Y x_s, \quad x_s = \mu L_+ - \beta Y x_s$$
$$L_1 = \chi R_1 + \chi' S_1 \quad (Y \equiv (\ln F)'_{x=0} = 2(M_+^2 - 1)M_+' / [2 + (\varkappa - 1)M_+^2]M_+)$$

with fairly arbitrary initial conditions for R, L, S, and x_s. In (1.1) R, L, and S are nonstationary perturbations of the left- and right-hand Riemann invariants and of the entropy function, respectively; U and A are stationary values of flow velocity and speed of sound; $M = U / A$ is the Mach number of stationary flow; $x = x_s(t)$ is the equation of the closing shock trajectory; coefficients a_{ij} are known functions of stationary parameters, hence of coordinate x; $\varphi, \psi, \varphi', \psi', \mu$, and β are known functions of M_- and of the adiabatic exponent $\varkappa$ of gas; subscripts "minus" and "plus" denote parameters for $x = 0$ on the left and right of the closing shock (gas flows from left to right), while subscript 1 denotes parameters at $x = 1$; χ and χ' are specified constants (coefficients of reflection)$\dot{x}_s = dx_s(t) / dt$; $F = F(x)$ is the cross-sectional area of the channel; $F' = dF / dx$ and $M' = dM / dx$. Expressions defining coefficients a_{ij}, $\varphi, \psi, \varphi', \psi', \mu$ and β appear in [2].

Since (1.1) is a system whose coefficients are independent of time, it admits solutions of the form

$$R(x, t) = R^\circ(x, \lambda)\exp\lambda t, \quad L(x, t) = L^\circ(x, \lambda)\exp\lambda t \tag{1.2}$$
$$S(x, t) = S^\circ(x, \lambda)\exp\lambda t, \quad x_s(t) = x_s^\circ \exp\lambda t$$

where λ are eigenvalues of related boundary problem (generally complex quantities), and $R^\circ(x, \lambda), \ldots$ are their corresponding eigenfunctions and the "amplitude" of shock oscillations (the magnitude of the latter is in the considered problem unimportant).

In problems of the considered kind eigenvalues form an infinite discrete sequence, while the system of eigenfunctions corresponding to these is not necessarily complete, which, generally speaking, does not allow expansions in functions (1.2) for solving the related mixed problem with arbitrary initial conditions. Examples of problems with equations of the hyperbolic kind with complete and incomplete systems of eigenfunctions are given in [6]. It is also shown there that even in cases of lack of completeness of related system the development of solution is determined by the extreme right-hand eigenvalue (in the complex plane λ). This aspect serves as the basis for using solutions (1.2) in stability investigations.

Equations and conditions that determine in (1.2) functions with superscript "o" are obtained by the substitution into (1.1) of appropriate expressions and reduction by $\exp \lambda t$. As the result we obtain the following system of ordinary differential equations and boundary conditions:

$$
\begin{aligned}
dR / dx &= [(a_{11} - \lambda) R + a_{12} L + a_{13} S] / (U + A) \\
dL / dx &= [a_{21} R + (a_{22} - \lambda) L + a_{23} S] / (U - A) \\
dS / dx &= - \lambda S / U \\
R_+ &= \varphi L_+ - \psi Y x_s, \quad S_+ = \varphi L_+ - \psi' Y x_s \\
\lambda x_s &= \mu L_+ - \beta Y x_s, \quad L_1 = \chi R_1 + \chi' S_1
\end{aligned}
\qquad (1.3)
$$

(here and subsequently the superscript "o" for the "complex amplitudes" is omitted).

In the considered problem the eigenvalues are those complex λ for which (1.3) admits nontrivial solutions. The disposition of eigenvalues is determined by the coefficients in equations and conditions (1.3) which, in turn, are uniquely related to the channel shape, the Mach number ahead of the closing shock, and the adiabatic exponent. When all eigenvalues λ lie in the left-hand half-plane, the flow is stable. If, however, the extreme right-hand eigenvalue has a positive real part, the flow is unstable. For elucidating the question of eigenvalues disposition in the complex plane we apply the method which, as already mentioned, is a modification of the known method of D-separation [4, 5].

If the channel shape, the Mach number M_- ahead of the closing shock (or velocity U_-), and the adiabatic exponent are fixed, the disposition of eigenvalues is completely determined by the reflection coefficients χ and χ'. We define $D(n)$ as the region of plane $\chi\chi$ in which n eigenvalues have positive real parts. Then for χ and χ contained in region $D(0)$ the flow is stable. Since the boundaries of regions $D(n)$ and $D(n-1)$ are represented by curves (the "Nyquist curves") that correspond to pure imaginary values of λ, i.e. $\lambda = i\omega$, where ω is a real number, the investigation of stability is to be carried out in two stages. First, by some method (e.g., on the basis of considerations of continuous passing to cases investigated in [1-3]) we determine in the $\chi\chi$-plane some point O that corresponds to a steady flow, i.e. which belongs to region $D(0)$. Then Nyquist curves are constructed for all $-\infty < \omega < +\infty$ and the smallest neighborhood of point O which is not reached by these curves, is determined. Since to the Nyquist curves correspond such χ and χ for which system (1.3) has nontrivial solutions, the construction of these curves can be effected by the method described below.

The three conditions in (1.1) that are satisfied for $x = 0$ and any ω make it possible to express R_+, L_+, and S_+ or their real or imaginary parts R_{r+}, R_{i+}, etc. in terms of x_s. We are interested here in nontrivial solutions, consequently, it is possible to substitute for x_s any arbitrary nonzero constant. Let us set $x_s = \mu$. Owing to the problem linearity, the substitution for the specified x_s of any other complex constant x_{s0} results in the multiplication of all results by the ratio x_{s0} / μ, which, as can be shown, is immaterial in what follows. Substituting into the related conditions in (1.3) $\lambda = i\omega$ and $x_s = \mu$, and separating in the obtained equalities the real and imaginary parts, we find that for $x = 0$

$$
R_{r+} = (\beta\varphi - \mu\psi) Y, \quad R_{i+} = \omega\varphi, \quad L_{r+} = \beta Y \qquad (1.4)
$$

$$L_{i+} = \omega, \quad S_{r+} = (\beta\varphi' - \mu\psi')Y, \quad S_{i+} = \omega\varphi'$$

The system of differential equations in (1.3) can be rewritten for $\lambda = i\omega$ as follows:

$$\begin{aligned} dR_r / dx &= (a_{11}R_r + a_{12}L_r + a_{13}S_r + \omega R_i) / (U + A) \\ dR_i / dx &= (a_{11}R_i + a_{12}L_i + a_{13}S_i - \omega R_r) / (U + A) \\ dL_r / dx &= (a_{21}R_r + a_{22}L_r + v_{23}S_r + \omega L_i) / (U - A) \\ dL_i / dx &= (a_{21}R_i + a_{22}L_i + a_{23}S_i - \omega L_r) / (U - A) \\ dS_r / dx &= \omega S_i / U, \quad dS_i / dx = -\omega S_r / U \end{aligned} \tag{1.5}$$

Integrating system (1.5) for a fixed ω from $x = 0$, where the unknown functions are defined by conditions (1.4), to cross section $x = 1$, we obtain in that section R_{r1}, R_{i1},. . . which are functions of ω. Substitution of these functions into the reflection conditions— the last equality in (1.3)—yields two equations

$$L_{r1}(\omega) = \chi R_{r1}(\omega) + \chi' S_{r1}(\omega), \quad L_{i1}(\omega) = \chi R_{i1}(\omega) + \chi' S_{i1}(\omega) \tag{1.6}$$

Solution of these equations for χ and χ' yields functions $\chi = \chi(\omega)$ and $\chi' = \chi'(\omega)$ which provide the parametric definition of the Nyquist curve. It will be readily seen that the described method of construction of the indicated formulas is some modification of the method developed in [7] for the determining frequency characteristics of an air intake channel.

Note that in this case, as in [1–3], a singular straight line corresponds to $\omega = 0$ [4, 5]. In fact, from (1.4) we have $R_{i+} = L_{i+} = S_{i+} = 0$ when $\omega = 0$. From this, in accordance with (1.5), follow the identities $R_i(x) \equiv L_i(x) = S_i(x) \equiv 0$ for $0 \leqslant x \leqslant 1$. Hence the second of Eqs. (1.5) is identically satisfied when $\omega = 0$, and the first is the equation of the singular straight line $L_{r1}(0) = \chi R_{r1}(0) + \chi' S_{r1}(0)$, where, as previously, $L_{r1}(0)$, $R_{r1}(0)$, and $S_{r1}(0)$ are quantities derived by integrating (1.5) with $\omega = 0$ or, rather, they represent subsystems of R_r, L_r, and S_r in (1.5) with appropriate initial conditions from (1.4).

The described method of constructing Nyquist curves is based on the integration of system (1.5) of linear equations with variable coefficients which, generally, can only be carried out numerically and for moderate ω does not present any difficulties (analysis shows that it is sufficient to restrict considerations to $\omega \geqslant 0$). However, the required computer time for integrating (1.5) ever increases with increasing ω. The point is that solution of (1.5) is of an oscillatory character whose period (with respect to x) is of order ω^{-1}. The integration step has to be, consequently, decreased in proportion to ω^{-1}, and this increases the computation time. This and the necessity to asymptotically analyze the behavior of Nyquist curves for $\omega \gg 1$ makes it important to obtain an analytic solution of system (1.5) which would hold for fairly considerable ω. This appears possible owing to the smallness of ω^{-1}. One of the possible ways of obtaining appropriate formulas obtained in [8] was used in [9], where, however, the simplifications due to the possibility of expressing the linearized equations of nonstationary flow in the characteristic form (1.1), were not taken into consideration. Allowance for this with its corollary, the use of Eqs. (1.3), substantially simplifies the derivation of solution. Thus the third of Eqs. (1.3) can be integrated independently of the first two, yielding

$$S(x,\lambda) = S_+(\lambda)\exp\left(-\lambda\int_0^x \frac{dx}{U}\right) \tag{1.7}$$

Here and in what follows $S_+(\lambda)$, as well as $R_+(\lambda)$ and $L_+(\lambda)$ are, in conformity with (1.3) $(x_s = \mu)$, expressed by

$$R_+(\lambda) = L(\lambda + \beta Y) - \psi Y\mu, \quad L_+(\lambda) = \lambda + \beta Y \tag{1.8}$$
$$S_+(\lambda) = \varphi'(\lambda + \beta Y) - \psi' Y\mu$$

To obtain the required formulas for $R(x,\lambda)$ and $L(x,\lambda)$ that would hold for $|\lambda| \gg 1$ and $|\operatorname{Re}\lambda| \sim 1$ we consider instead of the first two equations of (1.3) the more general system

$$dR/dx = [(a_{11} - \lambda)R + \varepsilon\lambda a_{12}L + a_{13}S]/(U + A) \tag{1.9}$$
$$dL/dx = [\varepsilon\lambda a_{21}R + (a_{22} - \lambda)L + a_{23}S]/(U - A)$$

where ε is a small parameter. When $\varepsilon = 1/\lambda$ system (1.9) reduces to the corresponding equations of (1.3).

If the solution of (1.9) with function $S(x,\lambda)$ defined by formula (1.7) is sought in the form of expansion in powers of ε, a system of "separable" linear differential equations which admit successive solution is obtained for the coefficients of related expansions. After necessary calculations and substitution in the derived formulas $\varepsilon = 1/\lambda$ and $x = 1$, we obtain, neglecting terms of higher order of smallness, the following formulas:

$$R_1(\lambda) \equiv R(1,\lambda) = r_0\exp(-\lambda\tau_r)\{R_+(\lambda) + \lambda^{-1}S_+(\lambda)[r_1 - r_2\exp(-\lambda\tau_r')] - \lambda^{-1}L_+(\lambda)(r_3 - r_4\exp\lambda\tau_{rl})\} \tag{1.10}$$
$$L_1(\lambda) \equiv L(1,\lambda) = l_0\exp\lambda\tau_l\{L_+(\lambda) + \lambda^{-1}S_+(\lambda)[l_1 - l_2\exp(-\lambda\tau_l')] - \lambda^{-1}L_+(\lambda)[l_3 - l_4\exp(-\lambda\tau_{rl})]\}$$
$$\tau_r = \int_0^1 \frac{dx}{A+U}, \quad \tau_l = \int_0^1 \frac{dx}{A-U}, \quad \tau_r' = \int_0^1 \frac{dx}{(A+U)M}$$
$$\tau_l' = \int_0^1 \frac{dx}{(A-U)M}$$
$$\tau_{rl} = \tau_r + \tau_l, \quad r_0 = \exp\int_0^1 \frac{a_{11}dx}{A+U}, \quad l_0 = \exp\int_0^1 \frac{a_{22}dx}{U-A}$$
$$r_1 = (Ma_{13})_+, \quad r_2 = (Ma_{13})_1/r_0, \quad r_3 = [(M-1)a_{12}/2]_+$$
$$r_4 = [(M-1)a_{12}]_1 l_0/2r_0, \quad l_1 = (Ma_{23})_+, \quad l_2 = (Ma_{23})_1/l_0$$
$$l_3 = [(M+1)a_{21}/2]_+, \quad l_4 = [(M+1)a_{21}]_1 r_0/2l_0$$

where, as previously, the subscripts "plus" and "unity" denote parameters at $x = 0$ and $x = 1$, respectively. The condition for $\operatorname{Re}\lambda$ formulated above ensures the boundedness of multipliers at λ^{-1} in the right-hand sides of (1.10) when $|\lambda| \gg 1$. In addition to these formulas we have in accordance with (1.7)

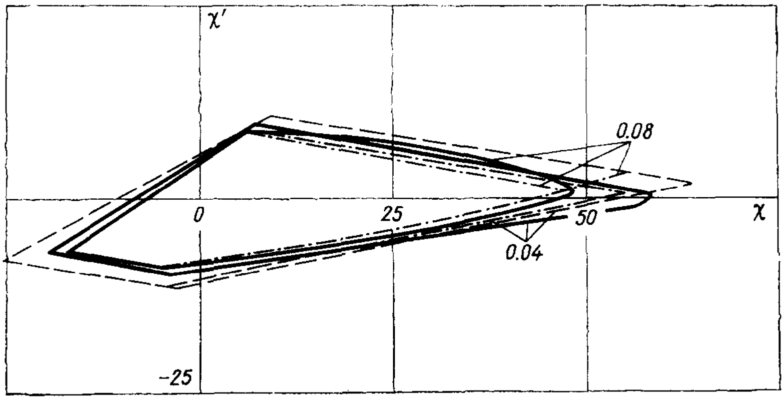

Fig. 1

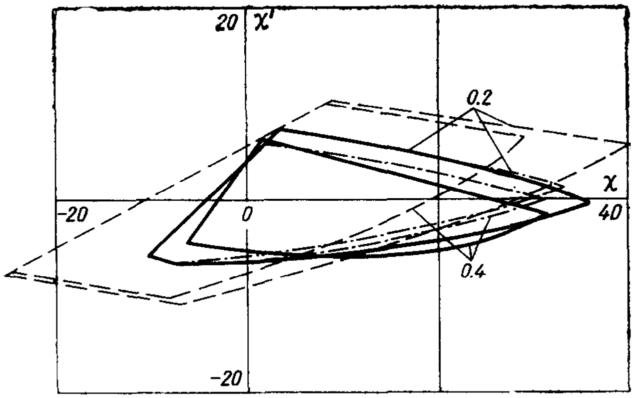

Fig. 2

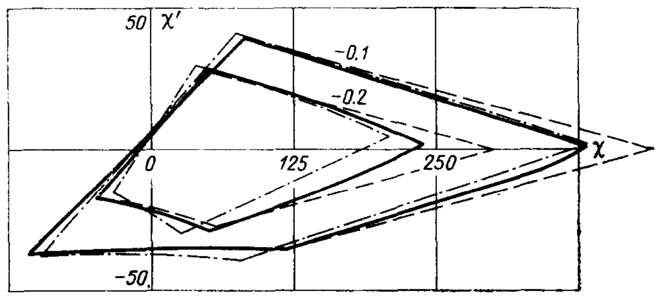

Fig. 3

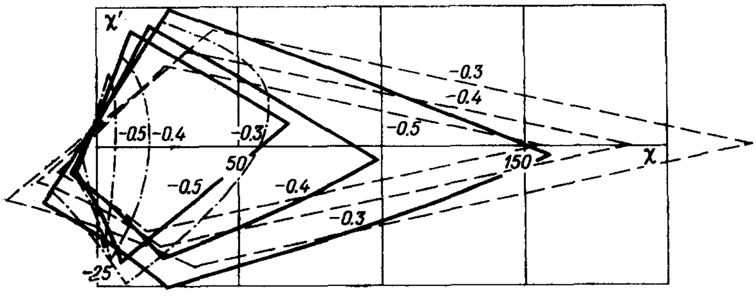

Fig. 4

$$S_1(\lambda) \equiv S(1, \lambda) = S_+(\lambda) \exp(-\tau_s \lambda) \quad \left(\tau_s = \int_0^1 \frac{dx}{U}\right) \tag{1.11}$$

Passing from numerical integration of (1.5) to formulas (1.10) and (1.11) in constructing Nyquist curves begins from such $\omega = \omega_* \gg 1$ for which the relative error of related formulas compared to numerical integration results is below some a priori established level. Note that the substitution of (1.10) and (1.11) with $R_+(\lambda)$, ... defined in (1.8) into the last of equalities (1.3) yields the characteristic equation of the considered problem, which is valid for those eigenvalues of λ for which $|\lambda| \gg 1$ when $|\operatorname{Re} \lambda| \sim 1$. As in [2,3], the analysis of the disposition of roots of such "asymptotic" characteristic equation is an obligatory element of stability investigation. Proceeding as in [2], it is possible to show that in the $\chi\chi$ - plane the stability region of the indicated equation is the rhombus

$$|\chi \varphi r_0 \pm \chi \varphi| < l_0 \tag{1.12}$$

The above exposition constitutes the basis for determining (in the $\chi\chi$ - plane) the stability region of any arbitrary stationary flow in a channel with a closing compression shock. As already noted, the position of a certain point O in region D (0) is determined by continuous passing to the results in [1–3], and the stability region is determined as the intersection of rhombus (1.12) with the minimal neighborhood of point O which is not reached by Nyquist curves determined by the method described above. In the examples considered below for $Y > 0$ point O always coincides with the coordinate origin ($\chi = \chi' = 0$).

2. The stability region boundaries determined in the $\chi\chi$ - plane by the method developed above are shown in Figs. 1–5 by solid lines. Calculations were carried out for $\varkappa = 1.4$.

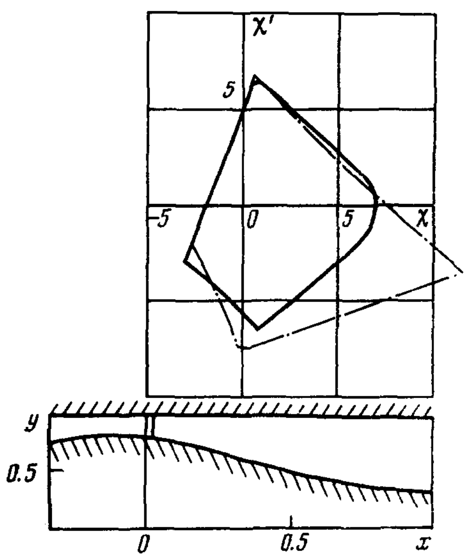

Fig. 5

In Figs. 1 and 2 these boundaries are plotted for $U = 1.34$ with the velocity related to the critical velocity of the stationary flow in the channel whose shape is such that

$$d \ln F(x) \,/\, dx = [d \ln F(x) \,/\, dx]_{x=0} = Y$$

Values of Y are indicated by figures at corresponding curves. Similar boundaries for the same values of Y shown in Figs. 1 and 2 by dash lines were determined with the use of the quasi-cylindrical approximation [2]. Solid and dash-line boundaries are very close for small Y but increasingly diverge with increasing Y, as expected, since the quasi-cylindrical approximation requires the fulfilment of condition $|Y| \ll 1$. The same tendency can be observed in the case represented in Figs. 3 and 4, where the dash-line boundaries were determined by the method of transonic approximation. These results relate to a channel shaped so that $M(x) = M_+ + M_+'x$ which means that the Mach number distribution of a stationary flow along the channel is linear. In the adduced examples $U_- = 1.1$ and M_+ varies from -0.1 to -0.5. Values of M_+ appear at the curves.

The last example relates to the flow in annalar channel whose cross section in the meridional plane xy of a cylindrical system of coordinates is shown in the lower part of Fig. 5 (dynamics of flow in such channel were numerically analyzed in [10]). The channel cross-sectional area between the shock plane $(x = 0)$ and the channel outlet increases by a factor of 2.33, and $U = 1.34$. Since the closing shock (the double line in Fig. 5) lies close to the minimum area cross section, the channel angle of opening at $x = 0$ is very small ($Y = 0.04$). However, this channel is by no means quasi-cylindrical since the quantity $d \ln F / dx$ "averaged" over the channel length is equal 0.86. This is also supported by the very pronounced difference of the stability region shown in Fig. 5 from the corresponding region ($Y = 0.04$) in Fig. 1.

In concluding our exposition we present the results of one more approximate method which we shall call the method of "frozen coefficients". It is based on the substitution of coefficients dependent on x of the first two of Eqs. (1.3) by constants. Since the obtained system with constant coefficients can be solved analytically, such method is very attractive. In spite of this, we must state explicitly that the formulation of conditions of its applicability is difficult. Thus, for example, the freezing of coefficients in the case of transonic approximation with $M_- \to 1$ apparently yields less accurate results than the theory developed in [3]. To estimate the accuracy of the indicated approach special computations in which the appropriate coefficients were taken as averages over the channel length were carried out. Results of these computations are shown in Figs. 1 - 5 by dash - dot lines.

The authors thank K. V. Brushlinskii for valuable consultation and V. A. Panin for his assistance in this work.

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Translated by J. J. D.